The surfer problem: A "whys" approach

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The class was accustomed to problem-based teaching.[1] So they were ready to work when they saw this classic problem (Jacobs, 1974, pp. 3-6; Jones and Shaw, 1988; Courant and Robbins, 1941, p. 359):

> A surfer, shipwrecked on an island in the shape of an equilateral triangle, wants to build a hut so that the sum of its distances to the three beaches is minimal. Where should the hut be located? (See Figure 1.)

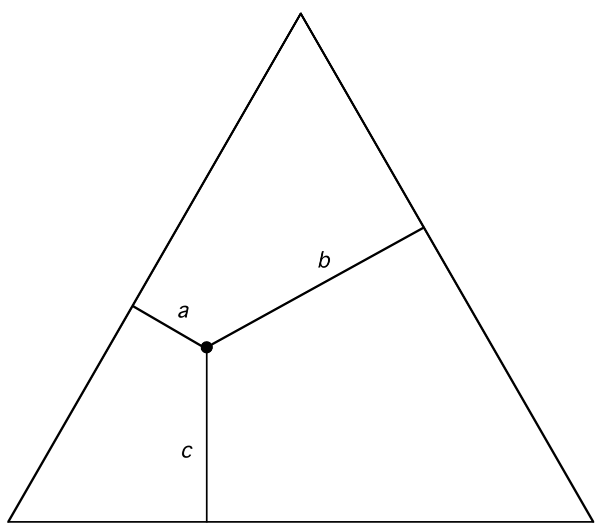

Figure 1: The surfer problem is to minimize $a + b + c$

The class members immediately looked around for materials. They found string, scissors, tape measures, protractors, and sticky notes. Some students wanted to work on Geometer's Sketchpad (Jackiw, 2001), but the instructor asked them not to. The centroid

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[1] In this article we compose experiences of classes of secondary teachers we have taught. That the problem is suitable for use with high school students is evidenced by its inclusion in Education Development Center (2000) and testimony of teachers who have used it, such as Cuoco, Goldenberg, and Mark (1995).

was a popular conjecture. In groups of four or five, the class members began to model the problem physically, constructing equilateral triangles (by SSS with 3 equal length strings or by SAS or ASA with a protractor), recording preliminary conjectures. As groups stuck notes at possible hut locations, made measurements, and added distances (recorded on the notes), they began to notice a pattern; their surprise at the results led them to make more measurements. Eventually, they generated a conjecture: The sum of distances is the same no matter where the hut is located. They didn't quite trust their conjecture, however, since their (imprecise) measurements varied so much. The students were not surprised that their instructor asked them to prove or disprove this conjecture for homework.

Why this pedagogy?

Some questions that might come to mind concern the teaching method used.

First, why didn't we pose the problem in a less time-consuming way? For example, we might have instructed,

> *Prove that the sum of distances from all points in the interior of an equilateral triangle to the three edges is the same.*

We chose our phrasing for several reasons. It is engaging for many students, with both a story and a kinesthetic component. Moreover, as Fawcett (1938) pointed out, the deepest understanding of a theorem (or non-theorem) can come when students know the "givens" but must explore and make conjectures about what is to be proved. Much later, Bell (1976) and de Villiers (1999) led the way in exploring the role of proof as providing an intellectual challenge for students as mathematicians. Observations that surprise students are a powerful stimulant for them to take on the challenge of proof.

Another question about the teaching approach could be why we asked students not to use geometry software initially. Our hunch is that student interest built more effectively over time this way than it would have with the quickness of automation. Moreover, we wanted to keep the physical setting active (rather than sedentary), and doing so also forced positive interdependence among group members, because they needed multiple people to stretch tape measures, measure, and track the points. Perhaps more importantly, measuring by hand led to interesting questions about errors: "Would more of those sums have been equal if we had measured more exactly?" To resolve such issues, students turned naturally to justification and proof.

First proofs

A number of proofs of the surfer conjecture have come from students, from colleagues, from the literature, or from us. We have arranged a selection of them here according to lessons we can learn from them.

Sketchpad proof?

The fact that Geometer's Sketchpad was not allowed at the outset did not prevent some of these students from presenting an electronic sketch, such as the one in Figure 2, as a proof in the next class session.

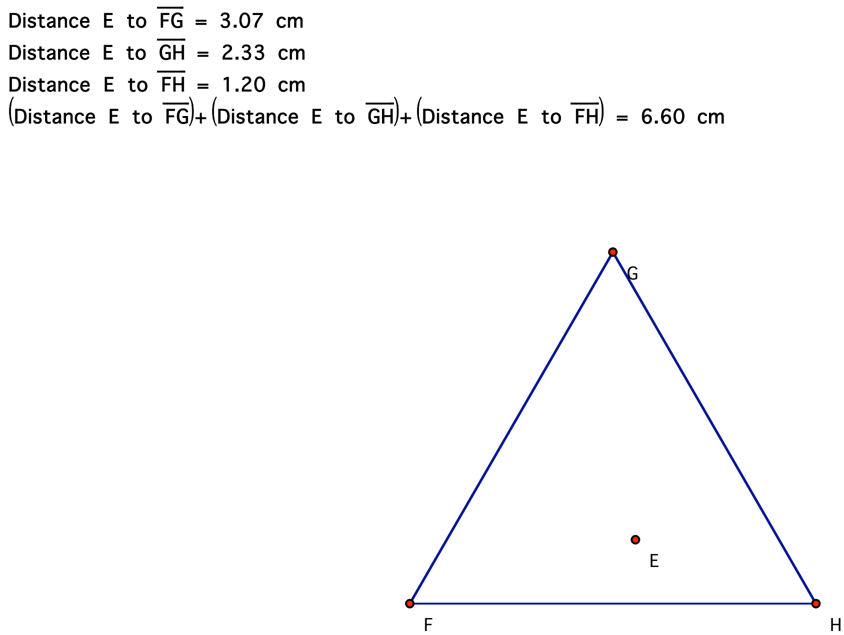

Figure 2: "Let's prove it in Sketchpad!"

Seeing that the sum of distances remained constant as the hut (point *E*) moved around convinced most of the students that the conjecture was true. Work with Sketchpad also convinced students that the triangle needs to be equilateral for the sum of distances to be constant, and that the moving point needs to be inside (or on) the triangle.

But was the demonstration a proof? The class began arguing. After all, it's not really a logical verification of the conjecture. Bell (1976) and de Villiers (1999) argue that proofs aren't just for verification; they can also serve the purpose of convincing. They can, but must they? Is every logically valid proof a convincing argument? Conversely, is every convincing argument a valid proof? If so, did the measurements done with string and tape measures constitute a proof?

When students "prove" empirically, whether with string or on Sketchpad, they do so in the everyday sense of the word. Some students extend to mathematics courtroom standards of "preponderance of evidence" or "beyond a reasonable doubt." Nonetheless, the mathematical standard for convincing that we try to grow in ourselves and our students goes beyond empiricism. We are not arguing that empiricism does not have a place in mathematics, but rather that it needs to be complemented by skepticism and a desire for logical proof.

The best proofs convince by giving insight into the mathematical relationships. Not only do they compel belief; they make the listener wiser. They do not just prove, they explain (Hanna 1995). They answer the question *Why?* The Sketchpad demonstration doesn't seem to answer the question *Why?* for most of us. We especially yearn for an answer to that question if, as in this case, a conjecture is counter-intuitive.

Coordinate geometry proof

Here's a sketch of a proof that is more logically compelling. Does it answer the question *Why?*

To minimize fractions, give the equilateral triangle's sides length $2s$ (with $s > 0$) and place its vertices at (0,0), ($2s$, 0), and (s, $\sqrt{3}$). (Figure 3)

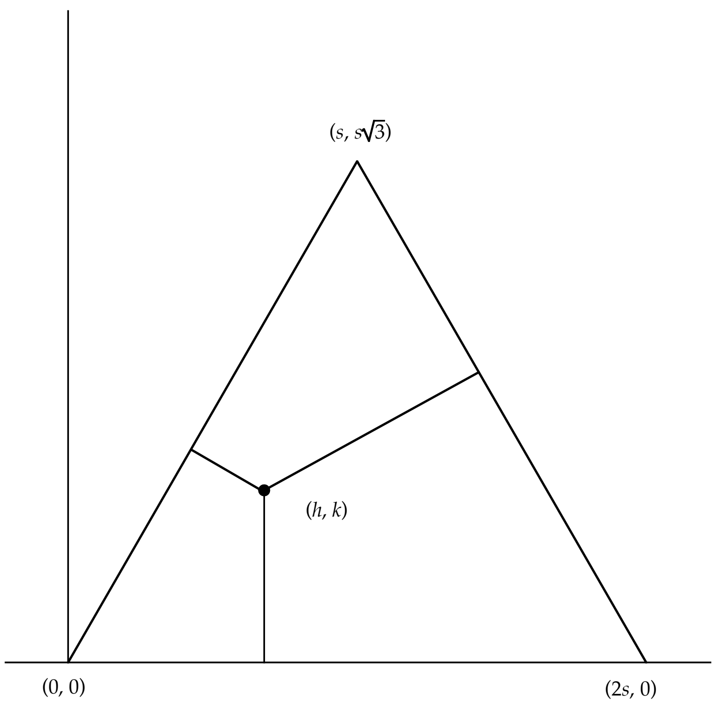

Figure 3: A coordinate geometry approach

The edges have equations $y = 0$, $y = \sqrt{3}\,x$, and $y = -\sqrt{3}\,x + 2\sqrt{3}\,s$. As needed, use the fact that the distance from point (h, k) to line $Ax + By + C = 0$ is $\left|\frac{Ah + Bk + C}{\sqrt{A^2 + B^2}}\right|$ to get the sum of distances from (h, k) to the edges as $\left|\frac{\sqrt{3}h - k}{\sqrt{4}}\right| + \left|\frac{-\sqrt{3}h - k + 2\sqrt{3}s}{\sqrt{4}}\right| + k$. Because the point ($h$, k) is inside the triangle, the absolute value signs can be dropped, making a sum of $\sqrt{3}\,s$, independent of the point (h, k). (In some students' work, it may be that the quantity can be shown to be negative, and a factor of –1 must be introduced to reach the same result.)

A question arises from this proof: Is it significant that the sum $\sqrt{3}\,s$ is the altitude of the equilateral triangle with side $2s$? This question may be an example of how, as de Villiers (1999) says, a proof might serve the purpose of discovery.

Area proof

Among the proofs we know of, this one requires the fewest words. (See Jones and Shaw, 1988.) Connect the hut with each vertex of the triangle. (Figure 4) Let the sides of this triangle have length *s.*

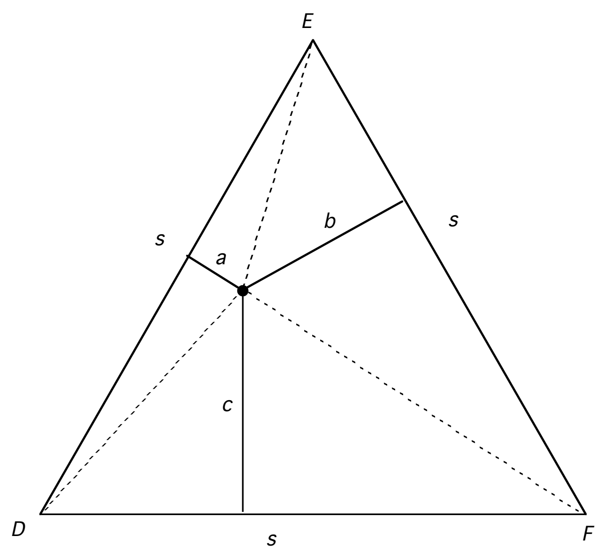

Figure 4: Area proof

These segments partition the triangle into the smaller triangles *DHE*, *FHE*, and *FHD*. The sum of the areas of these smaller triangles is $\frac{1}{2}as + \frac{1}{2}bs + \frac{1}{2}cs = \frac{1}{2}s(a + b + c)$.

That is, this sum is a constant times the sum of the three perpendiculars. But this sum is the area of the large triangle, so it is independent of the location of the hut. In fact, because the area of the equilateral triangle is $\frac{s^2\sqrt{3}}{4}$, we have that the constant sum is $a + b + c = \frac{s\sqrt{3}}{2}$, confirming the above calculation. One could then probe as suggested at the end of the coordinate geometry proof.

Why more than one proof?

We've shown two proofs of the surfer conjecture. Why might it be useful for students to see more than one proof of a theorem?

One reason, as Winicki-Landman (1998) points out, is that students can learn to appreciate the elegance of a proof only by seeing some that aren't so elegant. The area proof might be seen to be more elegant than the coordinate geometry proof, for example. Another reason to show students more than one proof is to allow them to see different ways of thought. For example, the terse proof above illustrates that area is a good invariant to consider. In our experience, in a typical class of teachers about one or two have the ingenuity to hit on the area proof, but many will diligently (with well-placed hints such as "Have you used the fact that the hut is on the island?") plug through the coordinate geometry one. (Still others attempt synthetic methods, which can succeed.) To give only the area proof and say "you should have been ingenious" would be unsatisfactory. And, as the National Council for Teachers of Mathematics contends in *Geometry from Multiple Perspectives,* multiple proofs can be mutually enhancing and reinforcing (Coxford, Burks, Giamati, and Jonik, 1991).

A particularly compelling reason for us to show multiple proofs is to answer the question *Why?* This is what Bell (1976) called the *illumination* function of proof.

We've come to realize that no single explanation answers the question *Why?* for everyone. Though most will be compelled to concede the truth of the conjecture by either the area or the coordinate geometry proof, whether either makes them feel wiser is a different matter. Some people would be satisfied with the coordinate geometry proof above. Others would think the area proof is enlightening. Still others wouldn't be satisfied that either of these proofs answer the question *Why?*.

Even more proofs

A transformational proof

One student in the class proposed a transformational proof. Cut the triangle through a point parallel to one side. Then move the cutoff piece through a rotation and translation—transformations that preserve lengths—to another side of a smaller equilateral triangle (see Figure 5).

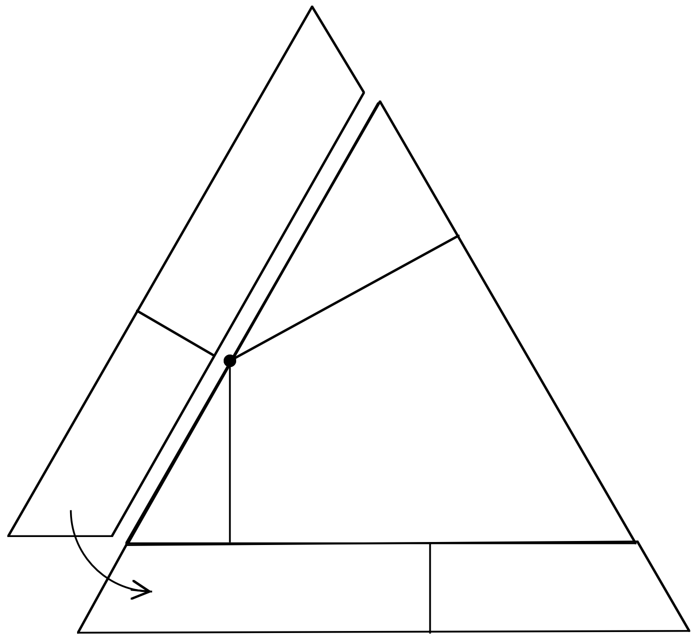

Figure 5: Transformation proof, step one

Cut the resulting triangle parallel to the third edge, making the special point a vertex of a smaller equilateral triangle. Rotate and translate the additional piece to make a triangle congruent to the original. (Figure 6)

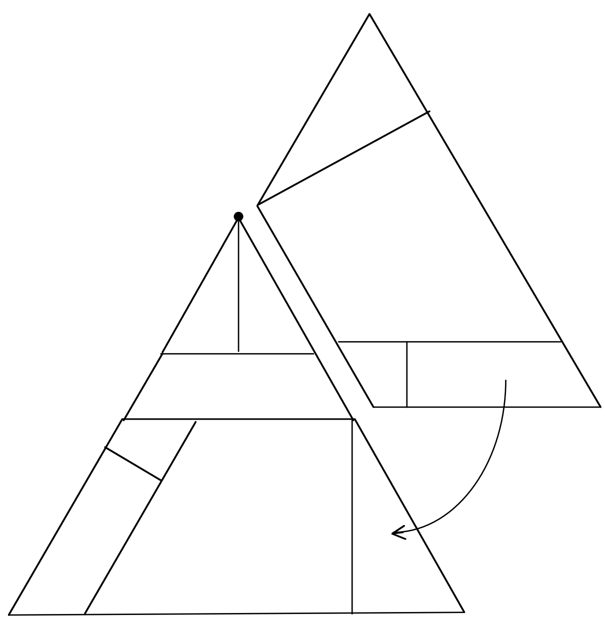

Figure 6: Transformation proof, step two

The lengths of the original paths to the beaches add up to an altitude of this triangle. So indeed the constant sum is an altitude, as we conjectured earlier.

In fact, one can construct a similar proof synthetically.

A right triangle proof

Some people prefer kinesthetic answers to the question *Why?* If the hut is at a vertex, then the sum of distances to the edges is the length of an altitude from that vertex. Now imagine moving the hut slightly away from that vertex. In the special case that the hut is still on the altitude, as in figure 7, the distance to the base along the altitude is shortened, but the distances to the other two sides are now non-zero.

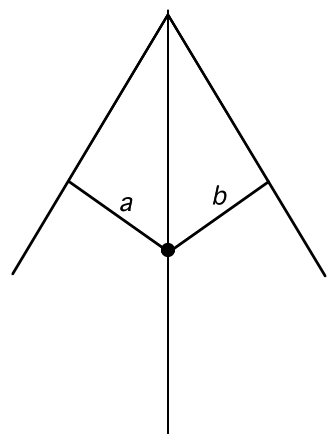

Figure 7: Invariance when the hut moves along the altitude

Has their sum compensated for what is lost in the distance to the base? Thanks to having two 30º-60º-90º triangles, each of the lengths *a* and *b* is half of the length lost, so the total distance is indeed invariant under motion in this special case.

In the more general case, where the hut isn't necessarily moved along the altitude (Figure 8), an auxiliary line parallel to the base makes a new equilateral triangle.

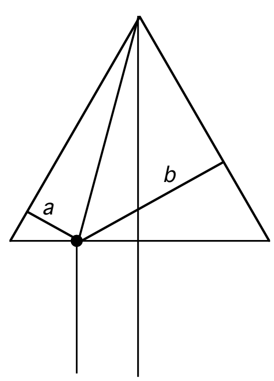

Figure 8: Invariance when the hut moves horizontally

Now a and b are legs of other 30-60 right triangles, each $\frac{\sqrt{3}}{2}$ times the corresponding hypotenuse. So the sum $a + b$ is $\frac{\sqrt{3}}{2}$ times the base of this smaller equilateral triangle and thus equals its altitude. Again, the sum of the distances is invariant under the displacement of the hut.

Earlier we mentioned several reasons that students benefit from seeing a variety of proofs. Another reason to vary the mathematical approach to the proof is to emphasize, with Dienes (1960), the common mathematical structure—at least what assumptions are being made. For example, it's not too difficult to see how the proofs above use the regularity of the triangle. Where does each proof use the assumption that the hut is in the interior of the island? (Jones and Shaw (1988) and Zheng (2002) consider, among other extensions, the relaxing of this assumption.) Proofs that give insight into mathematical structure illustrate Bell's (1976) *systematisation* function of proof.

Do you find any of these proofs better than any others? Are they more logically acceptable? Do they answer the question *Why?* for you?

As we look back over the proofs we've presented, we are practicing an important problem-solving strategy: reflection. Indeed, the proofs themselves illustrate quite a few such strategies: looking for invariants, thinking about special cases, reasoning by continuity, working backwards as well as forwards in proving, and the interplay of induction and deduction. Moreover, it is important for teachers to remember that these proofs are finished products, winnowed from a larger set. Demonstrating the messy process that created them and their uglier relatives invites students into mathematics as a human activity.

Whys reflections

Our primary reflection, however, has been considering what constitutes a satisfactory answer to the question *Why?* Brown (2001) has suggested that discussing questions such as *Which of these approaches is more elegant?* and even *Is it OK to discuss elegance in a mathematics class?* can help students to see that mathematics is a human activity and thus to grow in their understanding of humanity. We propose that similar benefits can accrue from a discussion of the question *What is a satisfactory answer to the question Why?*"

Human beings are faced with the question *Why?* in many aspects of their lives—social, aesthetic, religious. Pedagogical approaches that engage students in experiencing a variety of answers to *Why?* in the mathematical realm can help them learn about their own standards. A mathematics teacher's public recognition of the importance of the question can increase students' appreciation of mathematics as a human activity and of themselves as human beings.

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